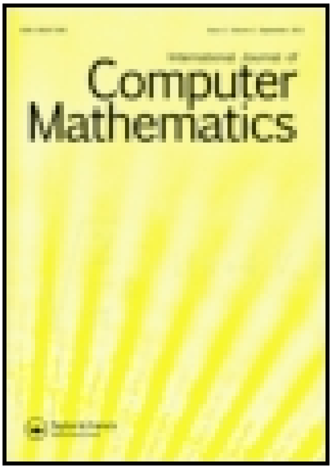

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THE SOLUTION OF THE SINE-GORDON EQUATION USING THE METHOD OF LINES

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The method of lines is used to transform the initial/boundary-value problem associated with the nonlinear hyperbolic sine-Gordon equation, into a first-order, nonlinear, initial-value problem.
Numerical methods are developed by replacing the matrix-exponential term in a recurrence relation by rational approximants. The resulting finite-difference methods are analysed for local truncation errors, stability and convergence. The results of a number of numerical experiments are given.

Keywords: Sine-Gordon equation; solution wave; method of lines; recurrence relation; rational approximants; global extrapolation.

C.R. Category: G.1.8.

1. INTRODUCTION

The sine-Gordon (SG) nonlinear hyperbolic equation has the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \sin u; \quad L_0 < x < L_1, \quad t > t_0, \tag{1}$$

The elucidation of equation (1) became important in physics with the evolution of the dislocation theory for crystals. Equation (1) also became important in

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connection with Josephson junction transmission lines, where sin$u$ is the Josephson current across an isolator between two superconductors, the voltage being proportional to $\partial u/\partial t$. Also, it is referred to as the Enneper equation and describes in differential geometry the variation of the angle u between asymptotic lines on surfaces of constant Gaussian curvature $K = -1$ under the assumption that the parameter lines for the description of the surface coincide with the curvature lines.

Initial conditions associated with the partial differential equation (PDE) given in (1) will be assumed to have the form

$$u(x,t_0) = f(x); \quad L_0 \leqslant x \leqslant L_1, \tag{2}$$

with initial velocity

$$\frac{\partial u}{\partial t}(x,t_0) = g(x) \quad L_0 \leqslant x \leqslant L_1. \tag{3}$$

Boundary conditions will be assumed to be of the form

$$\frac{\partial u(L_0,t)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u(L_1,t)}{\partial x} = 0; \quad t > t_0. \tag{4}$$

Equation (1) is a particular case of the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \frac{dV(u)}{du}, \tag{5}$$

where $dV(u)/du$ is a nonlinear function of u chosen as the derivative of a potential energy $V(u)$. Equation (5) occurs in a series of physical situations, as the propagation of waves in ferromagnetic materials carrying rotations of the direction of magnetization and of laser pulses in two-state media. For (1) $V(u) = 1 - \cos u$.

Steady travelling waves are obtained from (1) by putting

$$u(x,t) = U(x - ct) = U(J),$$

where c is a constant, which leads to the ordinary differential equation

$$\frac{1}{4}(1-c^2)\left(\frac{dU}{dJ}\right)^2 - \sin^2\left(\frac{U}{2}\right) = 0; \quad c^2 < 1, \tag{6}$$

and the constant of integration on the right-hand side of (6) is chosen to be zero. The first non-trivial solution of (6) is the solitary wave (soliton)

$$u(x,t) = 4\tan^{-1}\left[\pm\exp\left(\pm\frac{x-ct}{\sqrt{1-c^2}}+a\right)\right], \tag{7}$$

where a is an arbitrary constant and c indicates the velocity. The four possible sign combinations in (7) lead to kinks if the signs are similar, referred from now on as *kinks*$(+,+)$ or $(-,-)$ and to *antikinks* when the signs are opposite, referred from now on as antikinks $(+,-)$ or $(-,+)$. Figure 1 shows the corresponding soliton waves at times $t=9,18,108$ for $a=0$ and $c=0.5$.

2. NUMERICAL METHODS AND THEIR ANALYSES

Following Bratsos [2], to obtain a numerical solution, the so-called method of lines will be used: this method transforms the initial/boundary-value problem (IBVP) (1)–(4) into a second-order initial-value problem (IVP). To this effect the region $R=[L_0<x<L_1]\times[t>t_0]$ with its boundary ∂R consisting of the lines $x=L_0, x=L_1$ and $t=t_0$, is covered with a rectangular mesh, G_1^1, of points with coordinates $(x,t)=(x_m,t_n)=(L_0+mh, t_0+nl)$ with $m=0,1,\ldots,N,N+1$ and $n=0,1,\ldots$; clearly $h=(L_1-L_0)/(N+1)$. The solution of the SG equation at the typical mesh point (x_m,t_n) is $u(x_m,t_n)$ which may be denoted, when convenient, by u_m^n. The solution of an approximating difference scheme at the same point will be denoted by U_m^n; for the purpose of analysing stability, the numerical value of U_m^n actually obtained (subject, for instance, to computer round-off errors) will be denoted by $\tilde{U}_m^n$. Collectively, the values U_m^n will be written in vector form as

$$\mathbf{U}^n=\mathbf{U}(t)=[U_0^n, U_1^n,\ldots,U_{N+1}^n]^T;\quad n=0,1,2,\ldots, \tag{8}$$

T denoting transpose, so there are $N+2$ values of the solution to be determined at each time step.

The method of lines semi-discretizes the IBVP by replacing the space derivative in the SG equation (1) by the familiar second-order central-difference approximant

$$\frac{\partial^2 u(x,t)}{\partial x^2}=h^{-2}[u(x-h,t)-2u(x,t)+u(x+h,t)]+0(h^2) \text{ as } h\to 0. \tag{9}$$

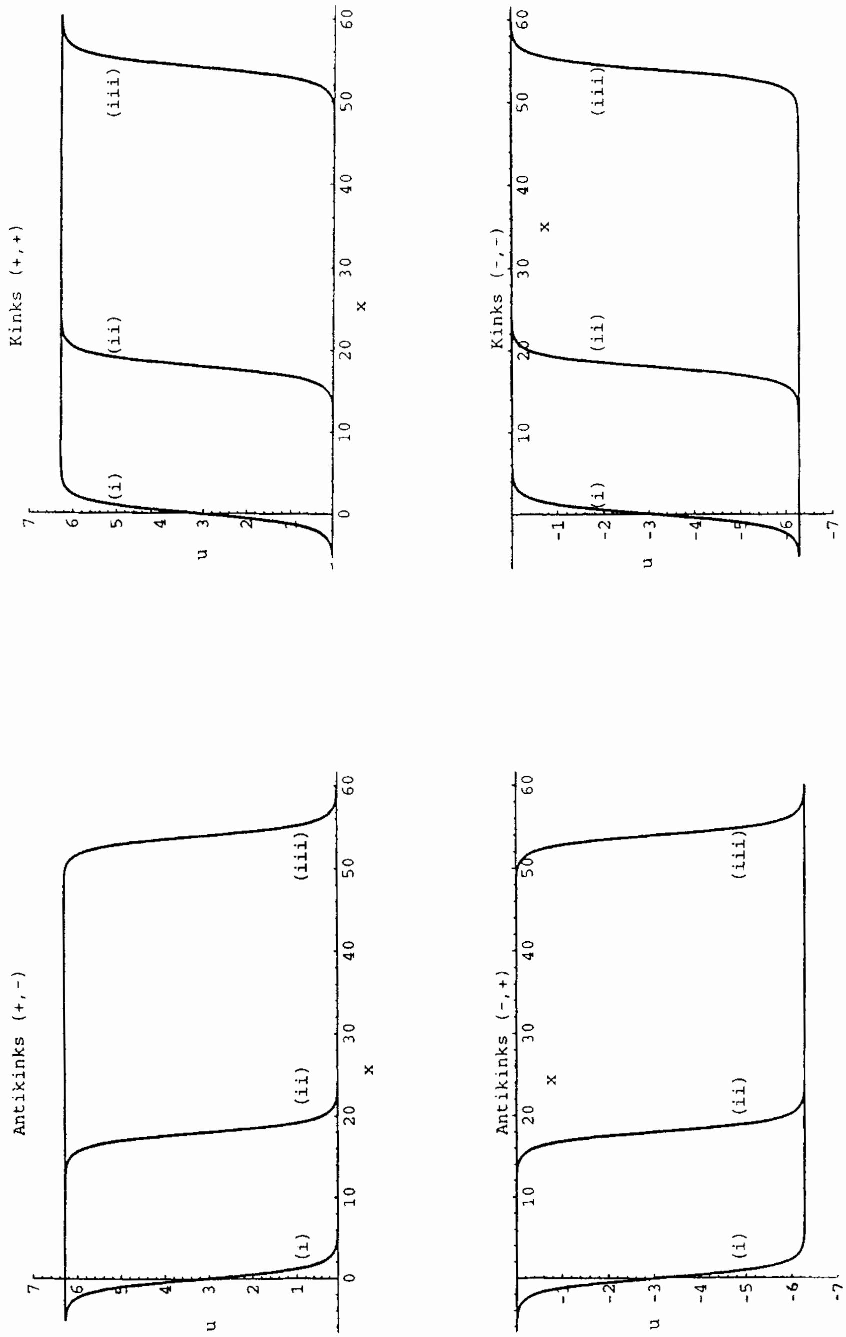

FIGURE 1 Soliton waves at times (i) $t = 9$, (ii) $t = 18$, (iii) $t = 108$ using $a = 0$ and $c = 0.5$.

Equation (1) with (9) is applied to all $N+2$ mesh points of G_1^1 at time level $t=t_n$ with $n=0,1,\ldots$. This, with the boundaries subject to (4) giving $U_{-1}^n = U_1^n$ and $U_{N+2}^n = U_N^n$, leads to an IVP of the form

$$D^2\mathbf{U}(t) = A\mathbf{U}(t) - \mathbf{G}(\mathbf{U}(t)), \quad t > t_0; \quad \mathbf{U}(t_0) = \mathbf{f}, \tag{10}$$

where $D^2 = \text{diag}\{d^2/dt^2\}$ is a diagonal matrix of order $N+2$,

$$\mathbf{G}^n = \mathbf{G}(\mathbf{U}(t_n)) = [\sin U_0^n, \sin U_1^n, \ldots, \sin U_{N+1}^n]^T$$

is a vector of order $N+2$ and A is the tri-diagonal matrix of order $N+2$ given by

$$A = h^{-2}\begin{bmatrix} -2 & 2 & & & & \\ 1 & -2 & 1 & & & \bigcirc \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ \bigcirc & & & 1 & -2 & 1 \\ & & & & 2 & -2 \end{bmatrix} \tag{11}$$

which has real, non-positive eigenvalues.

Using the relations

$$\mathbf{U}(t+l) = \exp(lD)\,\mathbf{U}(t)$$

$$\mathbf{U}(t-l) = \exp(-lD)\,\mathbf{U}(t)$$

where $D = \text{diag}\{d/dt\}$ is a matrix of order $N+2$, leads to the following three-time level recurrence relation for solving (1)

$$\mathbf{U}(t+l) = [\exp(lD) + \exp(-lD)]\,\mathbf{U}(t) - \mathbf{U}(t-l); \quad t = l, 2l, \ldots, \tag{12}$$

Numerical methods will be developed by replacing the matrix-exponential term in the recurrence relation (12), by rational approximants of the form

$$\exp(lD) \approx (I + a_1 lD + b_1 l^2 D^2)^{-1}(I + c_1 lD + d_1 l^2 D^2) \tag{13}$$

in such a way that there will be no need to evaluate $D\mathbf{U}(t)$.

Method I The (0, 2) Padé Approximant

Replacing the parameter values in (13) with $a_1 = b_1 = 0, c_1 = 1$ and $d_1 = 1/2$ equation (12) gives

$$\mathbf{U}(t+l) = (2I + l^2 D^2)\mathbf{U}(t) - \mathbf{U}(t-l) + 0(l^4) \tag{14}$$

with $D^2\mathbf{U}(t)$ given by (10). Let $r = l/h$, then applying (14) to the mesh point (x_m, t_n), gives the following five-point three-level explicit scheme

$$U_0^{n+1} = 2(1-r^2)\,U_0^n + 2r^2 U_1^n - l^2 \sin U_0^n - U_0^{n-1},$$

for $m = 0$,

$$U_m^{n+1} = 2(1-r^2)U_m^n + r^2(U_{m-1}^n + U_{m+1}^n) - l^2 \sin U_m^n - U_m^{n-1}, \tag{15}$$

for $m = 1, 2, \ldots, N$ and

$$U_{N+1}^{n+1} = 2r^2 U_N^n + 2(1-r^2)U_{N+1}^n - l^2 \sin U_{N+1}^n - U_{N+1}^{n-1}$$

for $m = N+1$.

The local truncation error of Method I arising from (14) is

$$L(x,t) = \frac{1}{12}\left(l^2 \frac{\partial^4 u}{\partial t^4} - h^2 \frac{\partial^4 u}{\partial x^4}\right) + \frac{1}{360}\left(l^4 \frac{\partial^6 u}{\partial t^6} - h^4 \frac{\partial^6 u}{\partial x^6}\right) + 0(l^6 + h^6), \tag{16}$$

where the first term on the right-hand side in (16) is the principal part, which tends to zero as $h, l \to 0$ simultaneously, so Method I is consistent with (1).

For the stability analysis of the method consider equation (15) written as

$$\frac{U_m^{n+1} - 2U_m^n + U_m^{n-1}}{l^2} = \frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{h^2} - \sum_{k=0}^{+\infty} (-1)^k \frac{(U_m^n)^{2k+1}}{(2k+1)!}, \tag{17}$$

where the last term on the right-hand side is Maclaurin's expansion of the term $\sin u$. Let $Z_m^n = U_m^n - \tilde{U}_m^n$, then

$$\frac{Z_m^{n+1} - 2Z_m^n + Z_m^{n-1}}{l^2} = \frac{Z_{m-1}^n - 2Z_m^n + Z_{m+1}^n}{h^2} - Z_m^n S_m^n, \tag{18}$$

where

$$S_m^n = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} [(U_m^n)^{2k} + (U_m^n)^{2k-1}\tilde{U}_m^n + \cdots + (\tilde{U}_m^n)^{2k}]$$

$$\approx \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} (U_G)^{2k} = \cos U_G,$$

after linearization of the term in square brackets, and U_G is a typical value of U_m^n for $m = 0, 1, \ldots, N+1$. Now let $Z_m^n = e^{\tilde{a}nl} e^{im\beta h}$ where $i = +\sqrt{-1}$, $\tilde{a}$ is a complex number and β is real. Then (18), after cancelling by $e^{\tilde{a}nl}e^{im\beta h}$, leads to the stability equation

$$\xi^2 - 2\left[1 - 2r^2 \sin^2\frac{\beta h}{2} - \frac{1}{2} l^2 \cos U_G\right]\xi + 1 = 0, \tag{19}$$

where $\xi = e^{\tilde{a}l}$ is the amplification factor. Equation (19) is a particular case ($\tilde{A} = 1$) of the following equation

$$\tilde{A}\xi^2 - 2\tilde{B}\xi + \tilde{A} = 0 \quad \text{with} \quad \tilde{A}, \tilde{B} \in \mathbb{R} \quad \text{and} \quad \tilde{A} \neq 0 \tag{20}$$

which will have roots ξ_1, ξ_2 with modulus less than or equal to unity for every ξ if

$$|\tilde{B}| \leqslant |\tilde{A}| \tag{21}$$

Condition (21) for (19) gives

$$r^2 \sin^2\frac{\beta h}{2} \geqslant -\frac{1}{4} l^2 \cos U_G \quad \text{and} \quad r^2 \sin^2\frac{\beta h}{2} \leqslant 1 - \frac{1}{4} l^2 \cos U_G.$$

The first inequality leads to

$$\frac{1}{h^2} \sin^2\frac{\beta h}{2} \geqslant -\frac{1}{4} \cos U_G, \tag{22}$$

and the second to

$$l^2\left(\frac{1}{h^2} \sin^2\frac{\beta h}{2} + \frac{1}{4} \cos U_G\right) \leqslant 1, \tag{23}$$

Now, (23) gives the following restriction for the time step

$$l \leqslant \left(\frac{1}{h^2}+\frac{1}{4}\right)^{-1/32}, \qquad (24)$$

Inequality (22) is always satisfied when $\cos U_G \geqslant 0$, while for $\cos U_G < 0$ it gives the following restriction for the space step

$$h \leqslant 2|\cos U_G|^{-1/2}. \qquad (25)$$

For the convergence analysis of the method consider equation (17), in which let $U_m^n = e^{in\psi}e^{im\theta}$, where ψ is a complex number and θ is a real number. Then equation (17), after cancelling both sides by $e^{in\psi}e^{im\theta}$, finally gives the following convergence equation,

$$\sin^2\frac{\psi}{2} = r^2\sin^2\frac{\theta}{2}+\frac{l^2}{4}\sum_{k=0}^{+\infty}(-1)^k\frac{(e^{in\psi}e^{im\theta})^{2k}}{(2k+1)!}. \qquad (26)$$

Then $|\sin^2\psi/2| \leqslant 1$, that is ψ is a real number, the last equation gives the following condition for convergence

$$l \leqslant \left[\frac{1}{h^2}+\frac{1}{4}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}\right]^{-1/2}. \qquad (27)$$

The condition imposed by (27) is more restrictive than that imposed by (24) and is therefore the condition to be used with (25).

Method II The (1, 1) Padé Approximant

Replacing the parameter values in (13) with $a_1 = -1/2, b_1 = d_1 = 0$ and $c_1 = 1/2$, equation (12) gives

$$\mathbf{U}(t+l)-\frac{l^2}{4}A\mathbf{U}(t+l)+\frac{l^2}{4}\mathbf{G}^{n+1} = 2\mathbf{U}(t)+\frac{l^2}{2}A\mathbf{U}(t)-\frac{l^2}{2}\mathbf{G}^{1}$$

$$-\mathbf{U}(t-l)+\frac{l^2}{4}A\mathbf{U}(t-1)-\frac{l^2}{4}\mathbf{G}^{n-1}, \qquad (28)$$

with $D^2\mathbf{U}(t)$ given by (10). Using a similar argument to that for Method I for the mesh point(x_m, t_n), equation (28) will give the following three-level, nine-point implicit scheme

$$\left(1+\frac{r^2}{2}\right)U_0^{n+1}-\frac{r^2}{2}U_1^{n+1}+\frac{l^2}{4}\sin U_0^{n+1}=(2-r^2)U_0^n+r^2U_1^n-\frac{l^2}{2}\sin U_0^n$$

$$-\left(1+\frac{r^2}{2}\right)U_0^{n-1}+\frac{r^2}{2}U_1^{n-1}-\frac{l^2}{4}\sin U_0^{n-1} \qquad (29)$$

for $m=0$,

$$-\frac{r^2}{4}(U_{m-1}^{n+1}+U_{m+1}^{n+1})+\left(1+\frac{r^2}{2}\right)U_m^{n+1}+\frac{l^2}{4}\sin U_m^{n+1}$$

$$=\frac{r^2}{2}(U_{m-1}^n+U_{m+1}^n)+(2-r^2)U_m^n$$

$$-\frac{l^2}{2}\sin U_m^n+\frac{r^2}{4}(U_{m-1}^{n-1}+U_{m+1}^{n-1})-\left(1+\frac{r^2}{2}\right)U_m^{n-1}-\frac{l^2}{4}\sin U_m^{n-1} \qquad (30)$$

for $m=1,2,\ldots,N$ and

$$-\frac{r^2}{2}U_N^{n+1}+\left(1+\frac{r^2}{2}\right)U_{N+1}^{n+1}+\frac{l^2}{4}\sin U_{N+1}^{n+1}=r^2U_N^n+(2-r^2)U_{N+1}^n-\frac{l^2}{2}\sin U_{N+1}^n$$

$$+\frac{r^2}{2}U_N^{n-1}-\left(1+\frac{r^2}{2}\right)U_{N+1}^{n-1}-\frac{l^2}{4}U_{N+1}^{n-1} \qquad (31)$$

for $m=N+1$.

This scheme forms the nonlinear algebraic system

$$\mathbf{F}(\mathbf{U}^{n+1})=\mathbf{0}. \qquad (32)$$

where the vector $\mathbf{U}(t_n+l)=\mathbf{U}^{n+1}$ can be determined by solving (32) either by fixed point iteration or by Newton's method. The Jacobian matrix J for Newton's method is a tri-diagonal symmetric matrix of order $N+2$.

The local truncation error of Method II is

$$L(x,t) = -\frac{1}{12}\left(2l^2\frac{\partial^4 u}{\partial t^4} + h^2\frac{\partial^4 u}{\partial x^4}\right)$$

$$-\frac{1}{720}\left(58l^4\frac{\partial^6 u}{\partial t^6} + h^4\frac{\partial^6 u}{\partial x^6}\right) + 0(l^6 + h^6), \tag{33}$$

so the method is consistent with (1).

Using a similar approach to that used for method I, it can be shown that the stability equation of method II is

$$\left[1 + r^2\sin^2\frac{\beta h}{2} + \frac{l^2}{4}\cos\hat{U}_G\right]\xi^2 - 2\left[1 - r^2\sin^2\frac{\beta h}{2} - \frac{l^2}{4}\cos\hat{U}_G\right]\xi$$

$$+\left[1 + r^2\sin^2\frac{\beta h}{2} + \frac{l^2}{4}\cos\hat{U}_G\right] = 0, \tag{34}$$

where $\cos\hat{U}_G = \Sigma_{k=0}^{+\infty}(-1)^k(\hat{U}_G)^{2k}/(2k)!$, with $\hat{U}_G$ a constant typical value of U_m^{n-1}, U_m^n and U_m^{n+1} for $m = 0, 1, \ldots, N+1$. Equation (34) is of the form (20), so condition (21) gives

$$-\left[1 + r^2\sin^2\frac{\beta h}{2} + \frac{l^2}{4}\cos\hat{U}_G\right] \leqslant 1 - r^2\sin^2\frac{\beta h}{2} - \frac{l^2}{4}\cos\hat{U}_G$$

$$\leqslant 1 + r^2\sin^2\frac{\beta h}{2} + \frac{l^2}{4}\cos\hat{U}_G, \tag{35}$$

The left-hand side of (35) is always satisfied, while the right-hand side leads to (22) and ultimately to (25).

The convergence equation of Method II is

$$\left(1 + r^2\sin^2\frac{\theta}{2}\right)\sin^2\frac{\psi}{2} = r^2\sin^2\frac{\theta}{2} + \frac{l^2}{8}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\{[e^{i(n+1)\psi}e^{im\theta}]^{2k}e^{i\psi}$$

$$+ 2[e^{in\psi}e^{im\theta}]^{2k} + [e^{i(n-1)\psi}e^{im\theta}]^{2k}e^{-i\psi}\}, \tag{36}$$

So, whenever $|\sin^2\psi/2| \leqslant 1$, that is ψ is a real number, the above equation gives

$$\left|\left(1-\frac{r^2}{4}\sin^2\frac{\theta}{2}\right)\sin^2\frac{\psi}{2}\right| \leqslant r^2 + \frac{l^2}{2}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!} \leqslant 1,$$

so the restriction for the time step is

$$l \leqslant \left[\frac{1}{h^2} + \frac{1}{2}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}\right]^{-1/2}. \tag{37}$$

Method III The (1,2) Padé Approximant

Replacing the parameter values in (13) with $a_1 = -1/3$, $b_1 = 0$, $c_1 = 2/3$ and $d_1 = 1/6$ equation (12) gives

$$\begin{aligned}\mathbf{U}(t+l) - \frac{l^2}{9}A\mathbf{U}(t+l) + \frac{l^2}{9}\mathbf{G}^{n+1} &= 2\mathbf{U}(t) + \frac{7}{9}l^2 A\mathbf{U}(t)\\ &- \frac{7}{9}l^2\mathbf{G}^n + \mathbf{U}(t-l) + \frac{l^2}{9}A\mathbf{U}(t-l) - \frac{l^2}{9}\mathbf{G}^{n-1},\end{aligned} \tag{38}$$

which, when applied to the mesh point (x_m, t_n) gives a scheme analogous to (31) and finally a nonlinear algebraic system of the form (32).

The local truncation error of Method III is

$$\begin{aligned}L(x,t) = &-\frac{1}{36}\left(l^2\frac{\partial^2 u}{\partial t^4} + 3h^2\frac{\partial^4 u}{\partial x^4}\right)\\ &-\frac{1}{1080}\left(7l^4\frac{\partial^6 u}{\partial t^6} + 3h^4\frac{\partial^6 u}{\partial x^6}\right) + 0(l^6 + h^6),\end{aligned} \tag{39}$$

so the method is consistent with (1).

The stability equation of Method III is

$$\begin{aligned}&\left[1 + \frac{4r^2}{9}\sin^2\frac{\beta h}{2} + \frac{l^2}{9}\cos\hat{U}_G\right]\xi^2 - 2\left[1 - \frac{14r^2}{9}\sin^2\frac{\beta h}{2} - \frac{7l^2}{18}\cos\hat{U}_G\right]\xi\\ &+\left[1 + \frac{4r^2}{9}\sin^2\frac{\beta h}{2} + \frac{l^2}{9}\cos\hat{U}_G\right] = 0,\end{aligned} \tag{40}$$

which is also of the form (20), so condition (21) gives

$$-\left[1+\frac{4r^2}{9}\sin^2\frac{\beta h}{2}+\frac{l^2}{9}\cos\hat{U}_G\right]\leqslant 1-\frac{14r^2}{9}\sin^2\frac{\beta h}{2}-\frac{7l^2}{18}\cos\hat{U}_G$$

$$\leqslant 1+\frac{4r^2}{9}\sin^2\frac{\beta h}{2}+\frac{l^2}{9}\cos\hat{U}_G. \tag{41}$$

Then the right-hand side of (41) leads to inequality (22) and ultimately to (25), while the left-hand side to the following restriction for the time step

$$l\leqslant\left[\frac{9}{5}\left(\frac{1}{h^2}+\frac{1}{4}\right)\right]^{-1/2}. \tag{42}$$

The convergence equation of Method III is

$$\left(1+\frac{4r^2}{9}\sin^2\frac{\theta}{2}\right)\sin^2\frac{\psi}{2}=r^2\sin^2\frac{\theta}{2}+\frac{l^2}{36}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\{[e^{i(n+1)\psi}e^{im\theta}]^{2k}e^{i\psi}$$

$$+7[e^{in\psi}e^{im\theta}]^{2k}+[e^{i(n-1)\psi}e^{im\theta}]^{2k}e^{-i\psi}\}. \tag{43}$$

So, whenever $|\sin^2\psi/2|\leqslant 1$, that is ψ is a real number, the above equation gives

$$\left|\left(1-\frac{r^2}{4}\sin^2\frac{\theta}{2}\right)\sin^2\frac{\psi}{2}\right|\leqslant r^2+\frac{l^2}{4}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)!}\leqslant 1,$$

which leads to the restriction (27). The condition in (27) is more restrictive than that in (42) and is therefore the condition to be used with (25).

3. GLOBAL EXTRAPOLATION

Let L_1^1 denote the global error of a convergent method at time $t=T<+\infty$. Then L_1^1 has, in general, the form

$$L_1^1=l^pe_1+l^{p+2}e_2+h^qk_1+h^{q+2}k_2+\cdots+l^{p+2v}e_{2v}+h^{q+2v}k_{2v}$$

$$+0(l^{p+2v+2}+h^{q+2v+2}) \tag{44}$$

where the quantities e_i, k_i for $i=1,2,\dots,v$ are independent of h,l and t. In what follows, only the case $p=q$ will be examined.

Global Extrapolation in Time

Suppose that the time interval of integration is divided into $\tau_1 n$ subintervals, where $\tau_1=2,3,\dots$, each of width l/τ_1 giving a discretization $G_1^{\tau_1}$ consisting of the $n\tau_1+1$ points $t_{m,1}=t_0+ml/\tau_1$ for $m=0,1,\dots,\tau_1 n$ while the discretization in space remains the same. The application of a convergent method to find the solution at the point $T=\tau_1 n$ of $G_1^{\tau_1}$, which, when $p\neq q$, gives the global error

$$L_1^{\tau_1}=\tau_1^{-p}l^p e_1+\tau_1^{-(p+2)}l^{p+2}e_2+\cdots+\tau_1^{-(p+2v)}l^{p+2v}e_{2v}$$

$$+h^q k_1+0(l^{p+2v+2}+h^{q+2}). \tag{45}$$

Using a similar argument, it is possible to define the discretization $G_1^{\tau_2}$, where $\tau_2=2,3,\dots$ and $\tau_2\neq\tau_1$, consisting of the $n\tau_2+1$ points $t_{m,2}=t_0+ml/\tau_2$ for $m=0,1,\dots,\tau_2 n$ and generally the discretization $G_1^{\tau_v}$, where $\tau_v=2,3,\dots$ and $\tau_v\neq\tau_i$ for every $i=1,2,\dots,v-1$ consisting of the $n\tau_v+1$ points $t_{m,v}=t_0+ml/\tau_v$ for $m=0,1,\dots,\tau_v n$. Consider the approximation

$$U_E=\alpha_v U_1^{\tau_v}+\cdots+\alpha_2 U_2^{\tau_2}+\alpha_1 U_1^{\tau_1}+[1-(\alpha_v+\cdots+\alpha_2+\alpha_1)]U_1^1, \tag{46}$$

and the associated global error

$$L_E=\alpha_v L_1^{\tau_v}+\cdots+\alpha_2 L_1^{\tau_2}+\alpha_1 L_1^{\tau_1}+[1-(\alpha_v+\cdots+\alpha_2+\alpha_1)]L_1^1$$

Then

$$L_E=\alpha_v(\tau_v^{-p}l^p e_1+\tau_v^{-(p+2)}l^{p+2}e_2+\cdots+\tau_v^{-(p+2v)}l^{p+2v}e_{2v}+h^q k_1)+\cdots$$

$$+\alpha_2(\tau_2^{-p}l^p e_1+\tau_2^{-(p+2)}l^{p+2}e_2+\cdots+\tau_2^{-(p+2v)}l^{p+2v}e_{2v}+h^q k_1)$$

$$+\alpha_1(\tau_1^{-p}l^p e_1+\tau_1^{-(p+2)}l^{p+2}e_2+\cdots+\tau_1^{-(p+2v)}l^{p+2v}e_{2v}+h^q k_1)$$

$$+[1-(\alpha_v+\cdots+\alpha_2+\alpha_1)](l^p e_1+l^{p+2}e_2+\cdots+l^{p+2v}e_{2v}+h^q k_1),$$

or

$$L_E = l^p[1-\alpha_1(1-\tau_1^{-p})-\alpha_2(1-\tau_2^{-p})-\cdots-\alpha_v(1-\tau_v^{-p})]e_1$$

$$+l^{p+2}[1-\alpha_1(1-\tau_1^{-(p+2)})-\alpha_2(1-\tau_2^{-(p+2)})-\cdots-\alpha_v(1-\tau_v^{-(p+2)})]e_2+\ldots$$

$$+l^{p+2v}[1-\alpha_1(1-\tau_1^{-(p+2v)})-\alpha_2(1-\tau_2^{-(p+2v)})-\cdots-\alpha_v(1-\tau_v^{-(p+2v)})]e_v+h^qk_1,$$

so the v terms on the right hand-side vanish when the following system holds

$$\left.\begin{aligned}(1-\tau_1^{-p})\alpha_1+\quad(1-\tau_2^{-p})\alpha_2+\cdots+(1-\tau_v^{-p})\,\alpha_v&=1\\ [1-\tau_1^{-(p+2)}]\alpha_1+[1-\tau_2^{-(p+2)}]\alpha_2+\cdots+[1-\tau_v^{-(p+2)}]\alpha_v&=1\\ \cdot\qquad\qquad\cdot\qquad\qquad\cdot\qquad&\\ [1-\tau_1^{-(p+2v)}]\alpha_1+[1-\tau_2^{-(p+2v)}]\alpha_2+\cdots+[1-\tau_v^{-(p+2v)}]\alpha_v&=1\end{aligned}\right\}$$

or

$$\left.\begin{aligned}\tau_{11}\alpha_1+\tau_{12}\alpha_2+\cdots+\tau_{1v}\alpha_v&=1\\ \tau_{21}\alpha_1+\tau_{22}\alpha_2+\cdots+\tau_{2v}\alpha_v&=1\\ \cdot\qquad\cdot\qquad\cdot&\\ \tau_{v1}\alpha_1+\tau_{v2}\alpha_2+\cdots+\tau_{vv}\alpha_v&=1\end{aligned}\right\}\qquad(47)$$

System (47) can be written in matrix-vector form as

$$\mathbf{T}\boldsymbol{\alpha}=\mathbf{i},\qquad(48)$$

$\boldsymbol{\alpha}=[\alpha_1,\alpha_2,\ldots,\alpha_v]^T$, $\mathbf{i}=[1,1,\ldots,1]^T$ and

$$\mathbf{T}=\begin{bmatrix}\tau_{11} & \tau_{12} & \cdots & \tau_{1v}\\ \tau_{21} & \tau_{22} & \cdots & \tau_{2v}\\ \cdot & \cdot & & \cdot\\ \tau_{v1} & \tau_{v2} & \cdots & \tau_{vv}\end{bmatrix},\qquad(49)$$

with det $(\mathbf{T})\neq 0$, so system (48) has a unique solution. This global extrapolation which uses the $v+1$ discretizations $G_1^1, G_1^{\tau_1},\ldots,G_1^{\tau_v}$ has produced an approximation U_E defined by (46) which is of order $p+2v+2$ in time.

Global Extrapolation in Space

Suppose now that the space interval is divided into $\lambda_1(N+1)$ subintervals each of width h/λ_1, where $\lambda_1 = 2, 3, \ldots$, giving the discretization $G^1_{\lambda_1}$, while the discretization in time remains the same. Then the solution at the point $T = t < +\infty$ of $G^1_{\lambda_1}$, when $p \neq q$, gives the global error

$$L^1_{\lambda_1} = l^p e_1 + \lambda_1^{-q} h^q k_1 + \lambda_1^{-(q+2)} h^{q+2} k_2 + \cdots + \lambda_1^{-(q+2v)} l^{q+2v} k_{2v}$$
$$+ 0(l^{p+2} + h^{q+2v+2}).$$

Using a similar argument as before, it is possible to define the discretization $G^1_{\lambda_2}$, where $\lambda_2 = 2, 3, \ldots$ with $\lambda_2 \neq \lambda_1$ and generally the discretization $G^1_{\lambda_v}$, where $\lambda_v = 2, 3, \ldots$ with $\lambda_v \neq \lambda_i$ for every $i = 1, 2, \cdots, v-1$. Consider the approximation

$$U_E = \gamma_v U^1_{\lambda_v} + \cdots + \gamma_2 U^1_{\lambda_2} + \gamma_1 U^1_{\lambda_1} + [1 - (\gamma_v + \cdots + \gamma_2 + \gamma_1)] U^1_1, \qquad (50)$$

and the associated global error,

$$L_E = \gamma_v L^1_{\lambda_v} + \cdots + \gamma_2 L^1_{\lambda_2} + \gamma_1 L^1_{\lambda_1} + [1 - (\gamma_v + \cdots + \gamma_2 + \gamma_1)] L^1_1,$$

where this global extrapolation which uses the $v+1$ discretizations $G^1_1, G^1_{\lambda_1}, \ldots, G^1_{\lambda_v}$ will produce an approximation U_E defined by (45) which will be of order $q + 2v + 2$ in space provided

$$\left.\begin{array}{r} (1-\lambda_1^{-q})\gamma_1 + (1-\lambda_2^{-q})\gamma_2 + \cdots + (1-\lambda_v^{-q})\gamma_v = 1 \\ [1-\lambda_1^{-(q+2)}]\gamma_1 + [1-\lambda_2^{-(q+2)}]\gamma_2 + \cdots + [1-\lambda_v^{-(q+2)}]\gamma_v = 1 \\ \cdot \qquad\qquad \cdot \qquad\qquad \cdot \qquad \\ [1-\lambda_1^{-(q+2v)}]\gamma_1 + [1-\lambda_2^{-(q+2v)}]\gamma_2 + \cdots + [1-\lambda_v^{-(q+2v)}]\gamma_v = 1 \end{array}\right\}$$

or

$$\left.\begin{array}{r} \lambda_{11}\gamma_1 + \lambda_{12}\gamma_2 + \cdots + \lambda_{1v}\gamma_v = 1 \\ \lambda_{21}\gamma_1 + \lambda_{22}\gamma_2 + \cdots + \lambda_{2v}\gamma_v = 1 \\ \cdot \qquad\quad \cdot \qquad\quad \cdot \qquad \\ \lambda_{v1}\gamma_1 + \lambda_{v2}\gamma_2 + \cdots + \lambda_{vv}\gamma_v = 1 \end{array}\right\}$$

which can be written in matrix-vector form as

$$\Lambda \boldsymbol{\gamma} = \mathbf{i}, \qquad (51)$$

where $\boldsymbol{\gamma}=[\gamma_1,\gamma_2,\ldots,\gamma_v]^T$, $\mathbf{i}=[1,1,\ldots,1]^T$ and

$$\Lambda=\begin{bmatrix}\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1v}\\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2v}\\ \cdot & \cdot & & \cdot\\ \lambda_{v1} & \lambda_{v2} & \cdots & \lambda_{vv}\end{bmatrix},$$

with det $(\Lambda)\neq 0$, so system (51) has a unique solution.

This is a generalization of the work by Twizell and Khaliq [4], who used $\tau=\lambda=2$.

Global Extrapolation in Both Time and Space

Suppose now that the interval of integration is divided into $n\mu_1$ subintervals each of width $1/\mu_1$, where $\mu_1=2,3,\ldots$ and the space interval into $\mu_1(N+1)$ subintervals each of width h/μ_1 giving the discretization $G^{\mu_1}_{\mu_1}$. The application of a convergent method to find the solution at the point $T=t<+\infty$ of $G^{\mu_1}_{\mu_1}$, which, when $p\neq q$, leads to the global error

$$L^{\mu_1}_{\mu_1}=\mu_1^{-p}l^pe_1+\mu_1^{-(p+2)}l^{p+2}e_2+\cdots+\mu_1^{-(p+2v)}l^{p+2v}e_{2v}$$
$$+\mu_1^{-q}h^qk_1+\mu_1^{-(q+2)}h^{q+2}k_2+\cdots+\mu_1^{-(q+2v)}l^{q+2v}k_{2v}$$
$$+0(l^{p+2v+2}+h^{q+2v+2}).$$

Similarly, it is possible to define the discretization $G^{\mu_2}_{\mu_2}$, where $\mu_2=2,3,\ldots$ with $\mu_2\neq\mu_1$ and generally the discretization $G^{\mu_v}_{\mu_v}$, where $\mu_v=2,3,\ldots$ with $\mu_v\neq\mu_i$ for every $i=1,2,\ldots,v-1$. Consider the approximation

$$U_E=\delta_vU^{\mu_v}_{\mu_v}+\cdots+\delta_2U^{\mu_2}_{\mu_2}+\delta_1U^{\mu_1}_{\mu_1}+[1-(\delta_v+\cdots+\delta_2+\delta_1)]U^1_1, \quad (52)$$

and the associated global error,

$$L_E=\delta_vL^{\mu_v}_{\mu_v}+\cdots+\delta_2L^{\mu_2}_{\mu_2}+\delta_1L^{\mu_1}_{\mu_1}+[1-(\delta_v+\cdots+\delta_2+\delta_1)]L^1_1.$$

This global extrapolation which uses the discretizations $G^1_1, G^1_{\lambda_1},\ldots,G^1_{\lambda_v}$ will produce an approximation U_E defined by (52) which will be of order

$q+2v+2$ in space, provided

$$\left.\begin{array}{r}(1-\mu_1^{-p})\delta_1+(1-\mu_2^{-p})\delta_2+\cdots+(1-\mu_v^{-p})\delta_v=1\\ [1-\mu_1^{-(p+2)}]\delta_1+[1-\mu_2^{-(p+2)}]\delta_2+\cdots+[1-\mu_v^{-(p+2)}]\delta_v=1\\ \cdot\qquad\qquad\cdot\qquad\qquad\cdot\\ [1-\mu_1^{-(p+2v)}]\delta_1+[1-\mu_2^{-(p+2v)}]\delta_2+\cdots+[1-\mu_v^{-(p+2v)}]\delta_v=1\end{array}\right\}$$

or

$$\left.\begin{array}{r}\mu_{11}\delta_1+\mu_{12}\delta_2+\cdots+\mu_{1v}\delta_v=1\\ \mu_{21}\delta_1+\mu_{22}\delta_2+\cdots+\mu_{2v}\delta_v=1\\ \cdot\qquad\cdot\qquad\cdot\\ \mu_{v1}\delta_1+\mu_{v2}\delta_2+\cdots+\mu_{vv}\delta_v=1\end{array}\right\}$$

which can be written in matrix-vector form as

$$\mathbf{M}\boldsymbol{\delta}=\mathbf{i}, \tag{53}$$

where $\boldsymbol{\delta}=[\delta_1,\delta_2,\ldots,\delta_v]^T$, $\mathbf{i}=[1,1,\ldots,1]^T$ and

$$\mathbf{M}=\begin{bmatrix}\mu_{11} & \mu_{12} & \cdots & \mu_{1v}\\ \mu_{21} & \mu_{22} & \cdots & \mu_{2v}\\ \cdot & \cdot & & \cdot\\ \mu_{v1} & \mu_{v2} & \cdots & \mu_{vv}\end{bmatrix},$$

with det $(\mathbf{M})\neq 0$, so system (53) has a unique solution.

4. NUMERICAL RESULTS

In order to include the soliton wave as it travels with velocity c (see also Fig. 1), the IBVP (1)–(4) was solved numerically using Methods I–III, with initial time $t_0=0$ and boundary lines $L_0=-2$, $L_1=58$. Two separate cases for the initial condition (2) were examined. The first with $U(x,0)=0$, $x\in(-2,58)$, and the second $U(x,0)=u(x,0)$, $x\in(-2,58)$, that is the numerical solution is equal to the theoretical solution for $t_0=0$. For the numerical solution at the first time step $t=l$, two separate approximations were examined. In the first, the approximation

$$u(x,t+l)=u(x,t)+l\frac{\partial u}{\partial t}+\cdots+\frac{l^4}{4!}\frac{\partial^4 u}{\partial t^4}+0(l^5)$$

was used which, for $t = t_0$, gives the solution vector $\mathbf{U}(t_1) \approx \mathbf{U}(t_0 + l)$, with $u(x, t_0) = f(x)$ and the partial derivatives obtained from (7). In the second, the numerical solution taken to be equal to the theoretical solution for $t = t_0 + l$, denoted from now on as $U(x, l) = u(x, l)$, was used. Integrating from time t_n to time t_{n+1}, the following three cases for the boundary conditions are considered

i) $u = 0$ for $x = L_0$ and $x = L_1$,
ii) $u = u(L_0, t_n)$ and $u = u(L_1, t_n)$, the theoretical solution at time $t = t_n$ and
iii) $u = U_0^n$ and $u = U_{N+1}^n$, the computed solution at time $t = t_n$.

It was deduced from the numerical results that the most accurate and convergent results for all the cases examined were obtained for the initial conditions $U(x, 0) = u(x, 0)$, $U(x, l) = u(x, l)$ and boundary conditions (ii).

Let the error, $e = e_n$, be the value of $u_m^n - U_m^n$ with maximum modulus $(m = 1, 2, \ldots, N)$ at time level $t = nl$ for $n = 0, 1, \cdots$ Let the corresponding. percentage relative error be $\varepsilon = \varepsilon_n = e_n \times 100/u_{m'}^n$, the mean value of the errors be $\tilde{e} = \tilde{e}_n = (\Sigma_{i=1}^n e_n)/n$ and let x_e denote the x-coordinate of the point at which $e = e_n$ occurs.

Method I

The IBVP (1)–(4) was solved for kinks $(+, +)$ and $(-, -)$ with the theoretical parameter a and the velocity c having the values $a = 0$ and $c = 0.5$. The results are given in Table I, from which the following may be deduced

i) equal errors, e, were obtained for both the above kinks. In Table I, their relative errors are denoted by $\varepsilon^{(+,+)}$ and $\varepsilon^{(-,-)}$.

TABLE I Results of method I for the kinks $(+, +)$ and $(-, -)$ with theoretical parameter $a = 0$ and velocity $c = 0.5$

h	l	t	e	x_e	$\varepsilon^{(+,+)}$	$\varepsilon^{(-,-)}$	$\tilde{e}$
0.1	0.01	18	0.485D−03	13.30	0.775D−02	−0.174D+01	0.475D−03
		36	0.526D−03	26.80	0.837D−02	−0.340D+03	0.490D−03
		108	0.369D−03	35.50	0.175D+08	−0.587D−02	0.457D−03
	0.001	18	0.487D−03	13.40	0.777D−02	−0.196D+01	0.475D−03
		36	0.526D−03	26.80	0.838D−02	−0.341D+03	0.491D−03
		108	0.369D−03	35.50	0.175D+08	−0.587D−02	0.457D−03
0.05	0.01	18	0.126D−03	13.40	0.202D−02	−0.508D+00	0.119D−03
		36	0.130D−03	26.75	0.208D−02	−0.796D+02	0.122D−03
		108	0.915D−04	35.45	0.459D+07	−0.146D−02	0.114D−03
		18	0.491D−05	13.40	0.785D−04	−0.193D−01	0.447D−05
0.01		36	0.835D+01	19.15	0.159D+03	−0.799D+01	0.646D+00
		108	0.924D+01	2.70	0.123D+29	−0.759D+03	0.853D+01

ii) from the experiments it was deduced that the accuracy was improved as the space step was refined with $h \in (0.01, 0.1]$, while, when the time step was refined, the accuracy was approximately the same (see also Figure 2, where the surface shows the errors e from time $t = 0$ to $t = 18$ for kinks $(+,+)$ with $a = 0, c = 0.5$, $h = 0.1$ and $l = 0.1$).

iii) accurate results were also obtained for the kinks $(+,+)$ for $h \in [2, 3]$ given in Table II.

The IBVP (1)–(4) was also solved for antikinks $(-,+)$ and $(+,-)$, with the same values of the theoretical parameter a and the velocity c, and the results are given in Table III, from which the following may be deduced

i) equal errors, e, were also obtained for both the above antikinks. Their relative errors will be denoted by $\varepsilon^{(-,+)}$ and $\varepsilon^{(+,-)}$,

ii) there was also an improvement in accuracy as the space step was refined, with $h \in (0.01, 0.1]$, which did not happen as the time step was refined.

To examine the effect of the constant a and of the velocity c on the numerical results, the IBVP (1)–(4) was solved using Method I for kinks $(+,+)$ to time $t = 36$. The results are given in Table IV where it may be seen that the method diverges when c tends to unity ("fast soliton"), while, on the other hand, when a increases, the method becomes more accurate.

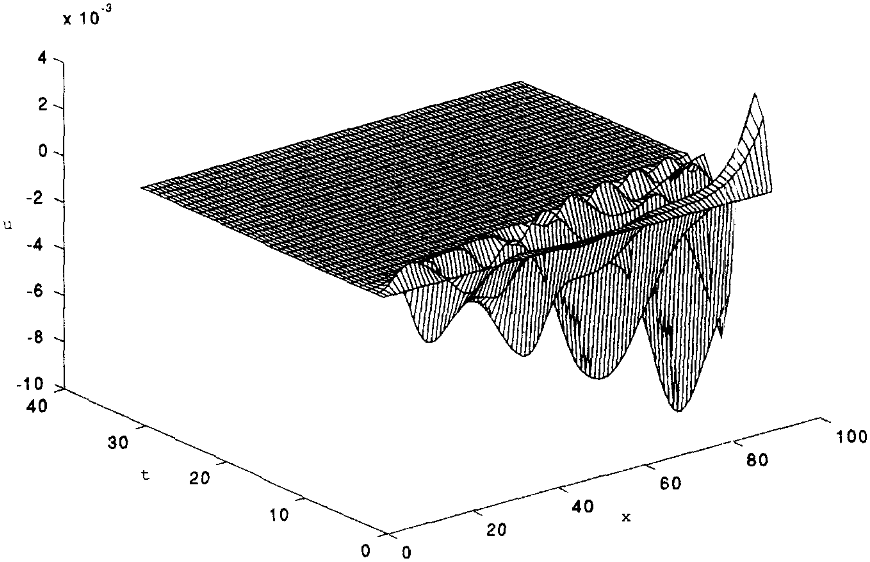

FIGURE 2 Errors for kinks $(+,+)$ using Method I with $a = 0$, $c = 0.5, h = 0.1$ and $l = 0.1$.

TABLE II Results of method I for the kinks $(+,+)$ with theoretical parameter $a=0$ and velocity $c=0.5$

h	l	t	e	x_e	$\varepsilon^{(+,+)}$	$\tilde{e}$
		18	0.749D−04	20.00	0.119D−02	0.196D+00
2	0.01	36	0.236D−04	30.00	0.376D−03	0.176D+00
		108	−0.395D+00	56.00	−0.671D+01	0.177D+00

TABLE III Results of method I for the antikinks $(-,+)$ and $(+,-)$ with parameter $a=0$ and velocity $c=0.5$

h	l	t	e	x_e	$\varepsilon^{(-,+)}$	$\varepsilon^{(+,-)}$	$\tilde{e}$
0.01	0.1	18	0.117D−01	9.20	−0.324D+00	0.434D+00	0.627D−02
		36	0.230D−01	18.10	−0.679D+00	0.787D+00	0.118D−01
		108	0.682D−01	54.00	−0.217D+01	0.217D+01	0.343D−01
	0.05	18	0.290D−02	9.20	−0.806D−01	0.108D+00	0.156D−02
		36	0.570D−02	18.10	−0.169D+00	0.196D+00	0.293D−02
		108	0.170D−01	54.05	−0.521D+00	0.561D+00	0.853D−02
		18	0.109D−03	9.19	−0.305D−02	0.402D−02	0.586D−04
	0.01	36	0.502D+00	3.57	−0.887D+02	0.799D+01	0.146D−01
		108	0.197D+02	55.09	−0.344D+10	0.178D+04	0.101D+02

TABLE IV Results of method I for the kinks $(+,+)$ with $h=0.05$, $l=0.01$ and $t=36$

c	a	e	x_e	ε	$\tilde{e}$
0	0	0729D−04	13.45	0.116D−02	0.111D−03
	30	0.125D−10	23.05	0.199D−09	0.119D−10
0.1	0	0.776D−04	13.45	0.124D−02	0.109D−03
0.3	0	0.991D−04	13.40	0.158D−02	0.109D−03
	30	0.125D−10	22.80	0.199D−09	0.119D−10
0.4	0	0.117D−03	13.35	0.187D−02	0.111D−03
0.5		0.126D−03	13.40	0.202D−02	0.119D−03
0.6		0.142D−03	17.35	0.226D−02	0.145D−03
0.8		0.596D−03	17.25	0.953D−02	0.321D−03
0.99		0.981D−01	16.60	0.140D+05	0.741D−01

The results of the global extrapolation method for the kinks $(+,+)$ subject to the stability and convergence restrictions are given in Table V. It is obvious that global extrapolation in space gives a significant improvement in accuracy when h is refined h/λ times, $(\lambda=2,3,\ldots,8)$ and this division gives zero remainder. From the experiments it is confirmed that the three types of global extrapolation give an improvement in accuracy of a numerical method, within intervals of h and/or l, if the non extrapolated results show an improvement in accuracy as h and/or l are refined in those intervals.

TABLE V Results of method I for the kinks (+, +) when $a=0$, $c=0.5, h=0.1, l=0.01$ at time $t=36$

Grids	e/ε	*Grids*	e/ε
G_1^1	0.52563D−03 0.837D−02	G_1^1, G_5^1	0.18255D−04 0.627D−03
G_1^1, G_1^5	0.52628D−03 0.838D−02	G_1^1, G_8^1	0.15698D−04 0.500D−03
G_1^1, G_4^1	0.20757D−04 0.773D−03	G_1^1, G_2^1, G_5^1	0.14271D−04 0.423D−03

TABLE VI Results of method II for the kinks (+, +) and (−, −) with $a=0$, $c=0.5$ and $H=2$

h	l	t	e	x_e	$\varepsilon^{(+,+)}$	$\varepsilon^{(-,-)}$	$\tilde{e}$
0.01	0.1	18	0.470D−03	14.80	0.749D−02	−0.953D+01	0.115D−02
		36	0.109D−02	17.30	0.651D−01	−0.237D−01	0.117D−02
		108	0.782D−03	53.20	0.517D−01	−0.164D−01	0.122D−02
		18	0.126D−03	8.50	0.614D−02	−0.296D−02	0.393D−04
	0.01	36	0.130D−03	17.40	0.703D−02	−0.295D−02	0.863D−04
		108	0.135D−03	53.45	0.694D−02	−0.312D−02	0.118D−03

TABLE VII Results of method II for the kinks (−, +) and (+, −) with $a=0$, $c=0.5$ and $H=2$

h	l	t	e	x_e	$\varepsilon^{(-,+)}$	$\varepsilon^{(+,-)}$	$\tilde{e}$
0.01	0.1	18	0.215D−02	9.50	−0.506D−01	0.105D+00	0.140D−02
		36	0.176D−02	18.70	−0.382D−01	0.105D+00	0.144D−02
		108	0.212D−02	54.60	−0.478D−01	0.114D+00	0.130D−02
		18	0.156D−03	9.60	−0.356D−02	0.849D−02	0.405D−04
	0.01	36	0.149D−03	18.55	−0.345D−02	0.767D−02	0.870D−04
		108	0.128D−03	54.55	−0.295D−02	0.657D−02	0.118D−03

TABLE VIII Results of method III for the kinks (−, −) with $a=0$, $c=0.5$ and $H=3$

h	l	t	e	x_e	$\varepsilon^{(+,+)}$	$\tilde{e}$
0.01	0.1	18	−0.120D−23	57.90	0.999D+02	−0.116D−24
		36	−0.391D−19	57.90	0.999D+02	−0.189D−20
		108	−0.442D−01	57.90	0.999D+02	−0.712D−03
		18	−0.113D−23	57.95	0.996D+02	−0.109D−24
	0.01	36	−0.368D−19	57.95	0.996D+02	−0.178D−20
		108	−0.416D−01	57.95	0.996D+02	−0.670D−03

Methods II–III

The IBVP (1)–(4) was solved using Methods II–III with the parameter values $a = 0$ and $c = 0.5$. The results are given in Tables VI–VII, in which H indicates the number of iterations necessary for the iterative method used to give accuracy of $M = 10^{-4}$; for starting value the theoretical value was used at each time level. It may be deduced from Tables VI–VII that

i) Method III has given inferior results to Method II. More precisely, Method III has a similar behaviour for the antikinks $(-, +)$, while for kinks $(+, +)$ and antikinks $(+, -)$ has given a numerical solution equal to zero,
ii) Method II shows an improvement in accuracy as the space step was refined with $h \in (0.01, 0.1]$, while that did not happen as the time step l was refined.

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