Non-Hermitian realization of a Lie-deformed Heisenberg algebra*

A. Jannussis a,b, A. Leodaris a,c, R. Mignani d,e

a Department of Physics, Patras University, 26110 Patras, Greece
b I.B.R., P.O. Box 1577, Palm Harbor, FL 34682-1577, USA
c General Department of Physics, Chemistry and Material Technology, Technological Educational Institutions of Athens, 12210 Egaleo, Greece
d Dipartimento di Fisica "E. Amaldi", III Università di Roma, Via Segrè 2, 00146 Rome, Italy
e I.N.F.N., Sezione di Roma I, c/o Dipartimento di Fisica, I Università di Roma "La Sapienza", P. le A. Moro 2, 00185 Rome, Italy

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Abstract

We discuss the non-Hermitian realization of a Lie-deformed, non-canonical Heisenberg algebra. We show that it essentially amounts to the case of a \( Q \)-deformed algebra with complex deformation parameter. The (real) energy eigenvalues of the corresponding oscillator are derived, whose deformed spectrum has, among the others, a ground state energy lower than that of the usual harmonic oscillator. The non-Hermitian deformed SU(2) algebra is also constructed.

Recently one of us (A.J.) has introduced [1,2] a new Lie-deformed Heisenberg algebra, defined by the commutation relations

\[
q_j(1 + i\lambda_{jk})p_k - p_k(1 - i\lambda_{jk})q_j = i\hbar\delta_{jk}, \tag{1}
\]

\[
[q_j, q_k] = 0, \quad [p_j, p_k] = 0 \tag{2}
\]

\((j,k=1,2,3)\), where \(q_j, p_j\) are the position and momentum operators and \(\lambda_{jk} = \lambda_k\delta_{jk}\), with \(\lambda_k\) real parameters. Obviously, for \(\lambda_{jk} = 0\) one recovers the usual Heisenberg algebra.

The new algebra possess some interesting properties [1,2]. First of all, the commutation relation (1) has a Lie-admissible [3] structure (whence the name of Lie-deformed Heisenberg algebra) because the left-hand side can be expressed in the form

\[
[q_j, p_k]_{LA} \equiv q_j T_{jk} p_k - p_k T_{jk}^* q_j, \tag{3}
\]

where

\[
T_{jk} = 1 + i\lambda_{jk}. \tag{4}
\]

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This allows one, among other things, to derive in a straightforward way the time-evolution of the operators $q, p$ [1] by means of the Santilli–Heisenberg formula [4]. Moreover, Eq. (1) can be rewritten as

$$[q_j, p_k] + i\lambda_{jk}\{q_j, p_k\} = i\hbar\delta_{jk}$$

(5)

(where $\{ , \}$ is the anticommutator, i.e. the (Lie-admissible) commutation relations of the new Lie-deformed algebra involve standard commutators and anticommutators.

Finally, the structure of the deformed algebra (1) is non-canonical, as can be easily seen by considering, for the sake of simplicity, a one-dimensional system and the Hamiltonian

$$H = \frac{1}{2}\omega(qp + pq).$$

(6)

In this case, indeed, Eq. (1) reduces to the noncanonical Heisenberg commutation relation [5,6]

$$[q, p] = i\hbar\left(1 - \frac{2\lambda}{\hbar\omega}H\right).$$

(7)

In the last years, there has been a renewed interest in noncanonical commutation relations [7–9] due to their link with quantum groups [7,8] and Lie superalgebras [9]. In this connection, let us notice that, for $A$ small, the commutation relation (1) in the one-dimensional case is – to first order in $\lambda$ – nothing but the $Q$-deformed commutator (Caldi’s “quommutator” [10])

$$qQp - pQ^{-1}q = i\hbar$$

with $Q = \exp(iA)$.

The operators $q_j$ and $p_k$ of the Lie-deformed Heisenberg algebra (1), (2) are obviously Hermitian, and their explicit expressions in the $q$- and $p$-representations read, respectively [1]

$$q_j \rightarrow q_j, p_k = \frac{\hbar}{2\lambda_k q_k} \left[1 - \exp(2i\theta_k q_k \partial/\partial q_k)\right],$$

(9)

$$p_k \rightarrow p_k, q_j = \frac{\hbar}{2\lambda_j p_j} \left[1 - \exp(-2i\theta_j p_j \partial/\partial p_j)\right],$$

(10)

where

$$\theta_k = \arctg \lambda_k.$$ (11)

Now we want to investigate if it is possible to get a realization of the algebra (1), (2) in terms of non-Hermitian operators. The answer is indeed positive.

To this aim, let us generalize Eqs. (1), (2) to the case of operators $A_j, B_k$ which are non-Hermitian (NH), i.e. ($\hbar = 1$)

$$A_j(1 + i\lambda_{jk})B_k - B_k(1 - i\lambda_{jk})A_j = i\delta_{jk},$$

(12)

$$[A_j, B_k] = 0 \quad (j \neq k), \quad [A_j, A_k] = 0, \quad [B_j, B_k] = 0$$

(13)

and

$$A_j^+(1 + i\lambda_{jk})B_k^+ - B_k^+(1 - i\lambda_{jk})A_j^+ = i\delta_{jk},$$

(14)

$$[A_j^+, B_k^+] = 0 \quad (j \neq k), \quad [A_j^+, A_k^+] = 0, \quad [B_j^+, B_k^+] = 0 \quad (A_j \neq A_j^+, B_k \neq B_k^+).$$

(15)

By using an extension of the bosonization method to the NH case [11], we seek the operators $A_j, B_k$ in the form

$$A_j = f_j(N_j + 1)a_j,$$ (16)
where \( a_j, a_j^+ \) are ladder operators of the usual Heisenberg-Weyl algebra, with \( N_j \) the corresponding number operator \( (N_j = a_j^+ a_j, N_j |n_j\rangle = n_j |n_j\rangle) \), and the structure functions \( f_j(N_j + 1) \) are complex. Then it is not difficult to get the following expressions of \( A_j \) and \( B_k \),

\[
A_j = \sqrt{\frac{i}{1 + i\lambda}} \left( \frac{[(1 - i\lambda_j)/(1 + i\lambda_j)]^{N_j+1} - 1}{1 - [(1 - i\lambda_j)/(1 + i\lambda_j)]} \right)^{1/2} a_j,
\]

\[
B_k = \sqrt{-i} \left( \frac{[(1 - i\lambda_k)/(1 + i\lambda_k)]^{N_k+1} - 1}{1 - [(1 - i\lambda_k)/(1 + i\lambda_k)]} \right)^{1/2}.
\]

As it easily follows from Eqs. (16)-(19), the operators \( A_j, B_k \) satisfy the relations

\[
(B_j^+)^+ = A_j, \quad (A_j^+)^+ = B_j,
\]

which can be regarded as the operator counterparts (in the Heisenberg representation) of the properties of states in the theory of non-Hermitian Hamiltonians [12].

For the sake of simplicity, and without any loss of generality, we can confine ourselves to the one-dimensional case.

The generalized number operator reads

\[
N = A^+ A = B B^+
\]

and from (18), (19) we find explicitly

\[
N = \frac{1}{\sqrt{1 + \lambda^2}} \left( 2 - \frac{[(1 + i\lambda)/(1 - i\lambda)]^N - [(1 - i\lambda)/(1 + i\lambda)]^N}{2 - (1 + i\lambda)/(1 - i\lambda) - (1 - i\lambda)/(1 + i\lambda)} \right)^{1/2}
\]

or

\[
N = \frac{1}{\sqrt{1 + \lambda^2}} \left| \frac{\sin(N\theta)}{\sin\theta} \right| n.
\]

where \( \theta = \arctg \lambda \) (cf. Eq. (11)). The operator \( N \) satisfies the condition

\[
N(\lambda) = N(-\lambda).
\]

Moreover, \( N \) is a Hermitian operator and therefore its eigenvalues are real [10]. The action of the operators \( A, A^+ \) and \( N(\lambda) \) on the kets \( |n\rangle \) is as follows,

\[
A|n\rangle = \sqrt{\frac{i}{1 + i\lambda}} \left( \frac{[(1 - i\lambda)/(1 + i\lambda)]^n - 1}{(1 - i\lambda)/(1 + i\lambda) - 1} \right)^{1/2} |n - 1\rangle,
\]

\[
A^+|n\rangle = \sqrt{-i} \left( \frac{[(1 + i\lambda)/(1 - i\lambda)]^{n+1} - 1}{(1 + i\lambda)/(1 - i\lambda) - 1} \right)^{1/2} |n + 1\rangle,
\]

\[
N(\lambda)|n\rangle = \frac{1}{\sqrt{1 + \lambda^2}} \left| \frac{\sin(n\theta)}{\sin\theta} \right| |n\rangle.
\]

Let us now consider the corresponding harmonic oscillator. In the usual Fock representation, i.e.,

\[
q = \sqrt{\hbar/2\omega} (A + A^+), \quad p = -i\sqrt{\hbar/2\omega} (A - A^+),
\]

(28)
the oscillator Hamiltonian takes the form

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 = \frac{1}{2} \hbar \omega (AA^* + A^*A)$$

(29)

and the (real) eigenvalues of the energy are given by

$$E_n = \frac{\hbar \omega}{2} \left[ \frac{1}{\sqrt{1 + \lambda^2}} \left| \sin \left( \frac{1}{2} (2n + 1) \theta \right) \right| \right] \sin \left( \frac{3}{2} \theta \right).$$

(30)

For $n = 0$ we get the ground state energy

$$E_0 = \frac{\hbar \omega}{2} \left( \frac{1}{\sqrt{1 + \lambda^2}} \right),$$

(31)

which is lower than the corresponding energy of the usual harmonic oscillator.

We finally consider the Lie-deformed SU(2) algebra in the non-Hermitian case. Let $A_j, A_j^+$ ($j = 1, 2$) be the boson operators of two independent, deformed oscillators with parameters $\lambda_j$, with explicit representation (18). The SU(2) generators are obtained as usual by the Jordan–Schwinger map

$$J_+ = A_1^+ A_2, \quad J_- = A_2^+ A_1, \quad 2J_z = [J_+, J_-].$$

(32)

After some algebra we get

$$2J_z = \cos \theta_1 \cos \theta_2 \frac{\sin(N_1 \theta_1) \sin[(N_2 + 1) \theta_2] - \sin[(N_1 + 1) \theta_1] \sin(N_2 \theta_2)}{\sin \theta_1 \sin \theta_2},$$

(33)

$$J_+ = \sqrt{\cos \theta_1 \cos \theta_2} \exp \left\{ \frac{1}{2} i \left( N_1 \theta_1 - (N_2 + 1) \theta_2 \right) \right\}$$

$$\times \sqrt{\frac{\sin(N_1 \theta_1) \sin[(N_2 + 1) \theta_2]}{\sin \theta_1 \sin \theta_2}} \frac{1}{\sqrt{N_1 (N_2 + 1)}} a_1^+ a_2,$$

(34)

$$J_- = \sqrt{\cos \theta_1 \cos \theta_2} \exp \left\{ \frac{1}{2} i \left( N_2 \theta_2 - (N_1 + 1) \theta_1 \right) \right\}$$

$$\times \sqrt{\frac{\sin(N_2 \theta_2) \sin[(N_1 + 1) \theta_1]}{\sin \theta_2 \sin \theta_1}} \frac{1}{\sqrt{N_2 (N_1 + 1)}} a_1 a_2^+,$$

(35)

with obvious action on the Fock space $|n_1, n_2\rangle$.

Summarizing, we have given a non-Hermitian realization of a Lie-deformed Heisenberg algebra which allows for real number-and oscillator-energy eigenvalues. The corresponding deformed oscillator has a spectrum whose features may be suitable for physical applications. Let us recall, in this connection, that deformed oscillators (like that introduced recently by one of us (A.J.) \cite{13}) have been shown to admit fruitful physical implications \cite{14}.

References


