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Non-Hermitian realization of a Lie-deformed Heisenberg algebra ★

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Abstract

We discuss the non-Hermitian realization of a Lie-deformed, non-canonical Heisenberg algebra. We show that it essentially amounts to the case of a Q-deformed algebra with complex deformation parameter. The (real) energy eigenvalues of the corresponding oscillator are derived, whose deformed spectrum has, among the others, a ground state energy lower than that of the usual harmonic oscillator. The non-Hermitian deformed SU(2) algebra is also constructed.

Recently one of us (A.J.) has introduced [1,2] a new Lie-deformed Heisenberg algebra, defined by the commutation relations

$$q_j(1+\mathrm{i}\lambda_{jk})p_k - p_k(1-\mathrm{i}\lambda_{jk})q_j = \mathrm{i}\hbar\delta_{jk}, \tag{1}$$

$$[q_j,q_k]=0, \qquad [p_j,p_k]=0 \tag{2}$$

$(j,k=1,2,3)$, where q_j, p_j are the position and momentum operators and $\lambda_{jk}=\lambda_k\delta_{jk}$, with λ_k real parameters. Obviously, for $\lambda_{jk}=0$ one recovers the usual Heisenberg algebra.

The new algebra possess some interesting properties [1,2]. First of all, the commutation relation (1) has a Lie-admissible [3] structure (whence the name of Lie-deformed Heisenberg algebra) because the left-hand side can be expressed in the form

$$[q_j,p_k]_{LA} \equiv q_jT_{jk}p_k - p_kT^{+}_{jk}q_j, \tag{3}$$

where

$$T_{jk} = 1+\mathrm{i}\lambda_{jk}. \tag{4}$$

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This allows one, among other things, to derive in a straightfoward way the time-evolution of the operators q, p [1] by means of the Santilli-Heisenberg formula [4]. Moreover, Eq. (1) can be rewritten as

$$[q_j, p_k] + i\lambda_{jk}\{q_j, p_k\} = i\hbar\delta_{jk} \tag{5}$$

(where $\{\ ,\ \}$ is the anticommutator), i.e. the (Lie-admissible) commutation relations of the new Lie-deformed algebra involve standard commutators and anticommutators.

Finally, the structure of the deformed algebra (1) is non-canonical, as can be easily seen by considering, for the sake of simplicity, a one-dimensional system and the Hamiltonian

$$H = \tfrac{1}{2}\omega(qp + pq). \tag{6}$$

In this case, indeed, Eq. (1) reduces to the noncanonical Heisenberg commutation relation [5,6]

$$[q, p] = i\hbar\left(1 - \frac{2\lambda}{\hbar\omega}H\right). \tag{7}$$

In the last years, there has been a renewed interest in noncanonical commutation relations [7-9] due to their link with quantum groups [7,8] and Lie superalgebras [9]. In this connection, let us notice that, for λ small, the commutation relation (1) in the one-dimensional case is - to first order in λ - nothing but the Q-deformed commutator (Caldi's "quommutator" [10])

$$qQp - pQ^{-1}q = i\hbar \tag{8}$$

with $Q = \exp(i\lambda)$.

The operators q_j and p_k of the Lie-deformed Heisenberg algebra (1), (2) are obviously Hermitian, and their explicit expressions in the q- and p-representations read, respectively [1]

$$q_j \to q_j, p_k = \frac{\hbar}{2\lambda_k}\frac{1}{q_k}[1 - \exp(2i\theta_k q_k \partial/\partial q_k)], \tag{9}$$

$$p_k \to p_k, q_j = \frac{\hbar}{2\lambda_j}\frac{1}{p_j}[1 - \exp(-2i\theta_j p_j \partial/\partial p_j)], \tag{10}$$

where

$$\theta_k = \operatorname{arctg} \lambda_k. \tag{11}$$

Now we want to investigate if it is possible to get a realization of the algebra (1), (2) in terms of non-Hermitian operators. The answer is indeed positive.

To this aim, let us generalize Eqs. (1), (2) to the case of operators A_i, B_k which are non-Hermitian (NH), i.e. ($\hbar = 1$)

$$A_j(1 + i\lambda_{jk})B_k - B_k(1 - i\lambda_{jk})A_j = i\delta_{jk}, \tag{12}$$

$$[A_j, B_k] = 0 \quad (j \neq k), \qquad [A_j, A_k] = 0, \qquad [B_j, B_k] = 0 \tag{13}$$

and

$$A_j^+(1 + i\lambda_{jk})B_k^+ - B_k^+(1 - i\lambda_{jk})A_j^+ = i\delta_{jk}, \tag{14}$$

$$[A_j^+, B_k^+] = 0 \quad (j \neq k), \qquad [A_j^+, A_k^+] = 0, \qquad [B_j^+, B_k^+] = 0 \qquad (A_j \neq A_j^+, B_k \neq B_k^+). \tag{15}$$

By using an extension of the bosonization method to the NH case [11], we seek the operators A_j, B_k in the form

$$A_j = f_j(N_j + 1)a_j, \tag{16}$$

$$B_k = a_k^+ f_k(N_k + 1), \tag{17}$$

where a_j, a_j^+ are ladder operators of the usual Heisenberg–Weyl algebra, with N_j the corresponding number operator ($N_j = a_j^+ a_j, N_j|n_j\rangle = n_j|n_j\rangle$), and the structure functions $f_j(N_j + 1)$ are complex. Then it is not difficult to get the following expressions of A_j and B_k,

$$A_j = \sqrt{\frac{\mathrm{i}}{1 + \mathrm{i}\lambda_j}} \left(\frac{[(1 - \mathrm{i}\lambda_j)/(1 + \mathrm{i}\lambda_j)]^{N_j+1} - 1}{(1 - \mathrm{i}\lambda_j)/(1 + \mathrm{i}\lambda_j) - 1} \frac{1}{N_j + 1} \right)^{1/2} a_j, \tag{18}$$

$$B_k = \sqrt{\frac{\mathrm{i}}{1 + \mathrm{i}\lambda_k}} a_k^+ \left(\frac{[(1 - \mathrm{i}\lambda_k)/(1 + \mathrm{i}\lambda_k)]^{N_k+1} - 1}{(1 - \mathrm{i}\lambda_k)/(1 + \mathrm{i}\lambda_k) - 1} \frac{1}{N_k + 1} \right)^{1/2}. \tag{19}$$

As it easily follows from Eqs. (16)–(19), the operators A_j, B_k satisfy the relations [11]

$$(B_j^*)^+ = A_j, \qquad (A_j^*)^+ = B_j, \tag{20}$$

which can be regarded as the operator counterparts (in the Heisenberg representation) of the properties of states in the theory of non-Hermitian Hamiltonians [12].

For the sake of simplicity, and without any loss of generality, we can confine ourselves to the one-dimensional case.

The generalized number operator reads

$$\mathcal{N} = A^+ A = BB^+ \tag{21}$$

and from (18), (19) we find explicitly

$$\mathcal{N} = \frac{1}{\sqrt{1 + \lambda^2}} \left(\frac{2 - [(1 + \mathrm{i}\lambda)/(1 - \mathrm{i}\lambda)]^N - [(1 - \mathrm{i}\lambda)/(1 + \mathrm{i}\lambda)]^N}{2 - (1 + \mathrm{i}\lambda)/(1 - \mathrm{i}\lambda) - (1 - \mathrm{i}\lambda)/(1 + \mathrm{i}\lambda)} \right)^{1/2} \tag{22}$$

or

$$\mathcal{N} = \frac{1}{\sqrt{1 + \lambda^2}} \left| \frac{\sin(N\theta)}{\sin\theta} \right|, \tag{23}$$

where $\theta = \operatorname{arctg} \lambda$ (cf. Eq. (11)). The operator $\mathcal{N}$ satisfies the condition

$$\mathcal{N}(\lambda) = \mathcal{N}(-\lambda). \tag{24}$$

Moreover, $\mathcal{N}$ is a Hermitian operator and therefore its eigenvalues are real [10]. The action of the operators A, A^+ and $\mathcal{N}(\lambda)$ on the kets $|n\rangle$ is as follows,

$$A|n\rangle = \sqrt{\frac{\mathrm{i}}{1 + \mathrm{i}\lambda}} \left(\frac{[(1 - \mathrm{i}\lambda)/(1 + \mathrm{i}\lambda)]^n - 1}{(1 - \mathrm{i}\lambda)/(1 + \mathrm{i}\lambda) - 1} \right)^{1/2} |n - 1\rangle, \tag{25}$$

$$A^+|n\rangle = \sqrt{\frac{-\mathrm{i}}{1 - \mathrm{i}\lambda}} \left(\frac{[(1 + \mathrm{i}\lambda)/(1 - \mathrm{i}\lambda)]^{n+1} - 1}{(1 + \mathrm{i}\lambda)/(1 - \mathrm{i}\lambda) - 1} \right)^{1/2} |n + 1\rangle, \tag{26}$$

$$\mathcal{N}(\lambda)|n\rangle = \frac{1}{\sqrt{1 + \lambda^2}} \left| \frac{\sin(n\theta)}{\sin\theta} \right| |n\rangle. \tag{27}$$

Let us now consider the corresponding harmonic oscillator. In the usual Fock representation, i.e.

$$q = \sqrt{\hbar/2m\omega}\, (A + A^+), \qquad p = -\mathrm{i}\sqrt{\tfrac{1}{2}\hbar m\omega}\, (A - A^+), \tag{28}$$

the oscillator Hamiltonian takes the form

$$H = p^2/2m + \tfrac{1}{2}m\omega^2 q^2 = \tfrac{1}{2}\hbar\omega(AA^+ + A^+A) \tag{29}$$

and the (real) eigenvalues of the energy are given by

$$E_n = \frac{\hbar\omega}{2}\frac{1}{\sqrt{1+\lambda^2}}\left|\frac{\sin[\frac{1}{2}(2n+1)\theta]}{\sin(\frac{1}{2}\theta)}\right|. \tag{30}$$

For $n = 0$ we get the ground state energy

$$E_0 = \frac{h\omega}{2}\frac{1}{\sqrt{1+\lambda^2}}, \tag{31}$$

which is lower than the corresponding energy of the usual harmonic oscillator.

We finally consider the Lie-deformed SU(2) algebra in the non-Hermitian case. Let A_j, A_j^+ $(j = 1,2)$ be the boson operators of two independent, deformed oscillators with parameters λ_j, with explicit representation (18). The SU(2) generators are obtained as usual by the Jordan–Schwinger map

$$J_+ = A_1^+ A_2, \qquad J_- = A_2^+ A_1, \qquad 2J_z = [J_+, J_-]. \tag{32}$$

After some algebra we get

$$2J_z = \cos\theta_1 \cos\theta_2 \frac{\sin(N_1\theta_1)\sin[(N_2+1)\theta_2] - \sin[(N_1+1)\theta_1]\sin(N_2\theta_2)}{\sin\theta_1 \sin\theta_2}, \tag{33}$$

$$J_+ = \sqrt{\cos\theta_1\cos\theta_2}\exp\{\tfrac{1}{2}\mathrm{i}[N_1\theta_1 - (N_2+1)\theta_2]\} \times\sqrt{\frac{\sin(N_1\theta_1)}{\sin\theta_1}\frac{\sin[(N_2+1)\theta_2]}{\sin\theta_2}}\frac{1}{\sqrt{N_1(N_2+1)}}a_1^+ a_2, \tag{34}$$

$$J_- = \sqrt{\cos\theta_1\cos\theta_2}\exp\{\tfrac{1}{2}[N_2\theta_2 - (N_1+1)\theta_1]\} \times\sqrt{\frac{\sin(N_2\theta_2)}{\sin\theta_2}\frac{\sin[(N_1+1)\theta_1]}{\sin\theta_1}}\frac{1}{\sqrt{N_2(N_1+1)}}a_1 a_2^+, \tag{35}$$

with obvious action on the Fock space $|n_1, n_2\rangle$.

Summarizing, we have given a non-Hermitian realization of a Lie-deformed Heisenberg algebra which allows for real number-and oscillator-energy eigenvalues. The corresponding deformed oscillator has a spectrum whose features may be suitable for physical applications. Let us recall, in this connection, that deformed oscillators (like that introduced recently by one of us (A.J.) [13]) have been shown to admit fruitful physical implications [14].

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