RESPONSE OF A COMPOSITE STENOSIS TO NON-NEWTONIAN BLOOD IN ARTERIES

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Abstract

The problem of blood flow through a composite stenosis in arteries has been investigated in the present work. To account for the non-Newtonian behavior, blood has been represented by a power-law fluid. The expression for the flow characteristics, namely, the impedance, the wall shear stress, the shear stress at the stenosis throat has been derived. We present some results concerning the dependence of these quantities on the geometrical parameters.

Key Words: Power-law, composite stenosis, impedance, shear stress, stenosis throat.

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INTRODUCTION

Cardiovascular diseases are known to be responsible in most of the cases of deaths and stenosis or arteriosclerosis (narrowing of any body passage, tube or orifice), is one of the common diseases. It is an unnatural and abnormal growth in the lumen of arterial wall segments, develops at various location of the cardiovascular system under the diseased conditions. Although the etiology of the initiation of the stenosis is not well understood, it has been suggested that the deposit of cholesterol on the arterial wall and proliferation of connective tissue may be responsible for the development of the disease. Regardless of the cause, it is well established that once the constriction has developed, it significantly affects the supply of the blood in arteries which may results into serious consequences (myocardial infraction, angina pectoris, cerebral strokes, etc.). The flow accelerates and consequently the velocity gradient near the wall region is steeper due to the increased core velocity resulting in relatively large shear stress on the wall even for a mild stenosis in the region of narrowing arterial constriction. With the knowledge that hemodynamic factors play an important role in the genesis and the proliferation of stenosis, this Area of research has attracted the attention of investigators. Since the first investigation of Mann et al. (1938), a large number of studies including the important contributions of Young(1968,1979), Young and Tsai (1973), Caro et al. (1978), Shukla et al.(1980), Ahmed and Giddens (1983), Sarkar and Jayaraman (1998), Pralhad and Schultz (2004), Jung et al. (2004), Liu et al. (2004) Srivastava and coworkers (1996, 2009, 2010a,b,c), Mishra et al. (2006), Misra and Verma (2007), Ponalagusamy (2007), Layek et al. (2005, 2009), Joshi et al. (2009), Mekheimer and El-Kot (2008), Tzirtzilakis (2008), Mandal and coworkers (2005, 2007a,b), Politis et al. (2007, 2008), Singh et al. (2010), Medhavi (2011) and many others; have been conducted in the literature in various context.

Being a suspension of corpuscles, blood behaves like a non-Newtonian fluid at low shear rate in small blood vessels (Hershey et al., 1964; Merill et al., 1965; Charm and Kurland, 1974). In particular, it has been pointed out that the flow behavior of blood
in small diameter tubes (less than 0.2mm) at less than 20 sec\(^{-1}\) shear rate, can be reasonably represented by a power-law fluid (Hershey et al., 1964; Charm et al., 1965; Huckaba and Hahn, 1968). In addition, a survey of literature on stenotic development indicates that most of the studies have been conducted are concerned with a single symmetric or non-symmetric stenoses. However, the recent observations Joshi et al., 2009; Srivastava and Rastogi, 2010a) reveal that stenoses may develop in series (multiple stenoses) or may be of irregular shapes or overlapping or of composite in nature. The present research is therefore devoted to study the effects of a composite stenosis on blood flow characteristics assuming that the flowing blood is represented by a power law fluid.

**FORMULATION OF THE PROBLEM**

Consider the axisymmetric flow of blood through a composite stenosis in an artery of circular cross-section. The geometry of the composite stenosis, assumed to be manifested in the arterial wall segment is described (Fig.1) as

\[
\frac{R(z)}{R_0} = 1 - \frac{2\delta}{R_0 L_0} (z - d); \quad d \leq z \leq d + L_0/2,
\]

\[
= 1 - \frac{\delta}{2R_0} \left\{ 1 + \cos \frac{2\pi}{L_0} (z - d - L_0/2) \right\}; \quad d + L_0/2 \leq z \leq d + L_0,
\]

\[
= 1; \quad \text{otherwise,}
\]

where \( R \equiv R(z) \) and \( R_0 \) are respectively, the radius of the artery with and without stenosis, \( L_0 \) is the length of the stenosis and \( d \) indicates its location, \( \delta \) is the maximum projection in the lumen.

Fig. 1 Geometry of a composite stenosis in an artery.

located at \( z = d + L_0/2 \).

Blood is assumed to be represented by a non-Newtonian (power-law) fluid. Following the reports (Young, 1968; Srivastava, 1995) and considering laminar, steady, one dimensional flow of blood in an artery, the general constitutive equation in a mild stenosis case \( \delta \ll R_0 \), may be written as

\[
\frac{dp}{dz} = -\frac{1}{r} \frac{\partial}{\partial r} (r \tau),
\]

where \( r \) is the radial coordinate measured normal to the tube axis and \( p \) is the pressure. The shear stress, \( \tau \) for a power-law fluid is given by
\[-\frac{du}{dr} = \left(\frac{\tau}{m}\right)^{1/n}, \quad (5)\]

where \(u\) is the axial velocity of the fluid, \(m\) is the consistency, and \(n\) is the flow behavior index (when \(n = 1\), then \(m = \mu\) is the Newtonian viscosity of the fluid).

The boundary conditions are

\[
\frac{\partial u}{\partial r} = 0 \text{ at } r = 0. \quad (7)
\]

**ANALYSIS**

Using equation (5) into (4), the solution of equation (4) under the boundary conditions (6) and (7), yields the expression for velocity, \(u\) as

\[
u = -\frac{n}{n+1} \left( -\frac{1}{2m} \frac{dp}{dz} \right)^{1/n} \left( R^{(n+1)/n} - r^{(n+1)/n} \right), \quad (8)
\]

The volumetric flow rate \(Q\) is now calculated as

\[
Q = 2\pi \int_0^R ru \, dr
\]

\[
= \frac{n \pi}{3n+1} \left( -\frac{1}{2m} \frac{dp}{dz} \right)^{1/n} R^{(3n+1)/n}, \quad (9)
\]

which yields

\[
\frac{dp}{dz} = -2m \left( \frac{3n+1}{n \pi} Q \right)^n \frac{1}{R^{3n+1}}. \quad (10)
\]

The pressure drop, \(\Delta p\) (= \(p\) at \(z = 0\) and \(-p\) at \(z = L\)) across the stenosis in the tube of length, \(L\) is calculated as

\[
\Delta p = \int_0^L \left( -\frac{dp}{dz} \right) \, dz
\]

\[
= \frac{2m}{R_0^{3n+1}} \left( \frac{3n+1}{n \pi} Q \right)^n \psi, \quad (11)
\]

where

\[
\psi = \int_0^d [\phi(z)]_{R/R_0 = 1} \, dz + \int_0^{d+L_d/2} [\phi(z)]_{R/R_0 \text{ from } (1)} \, dz + \int_0^{d+L_d} [\phi(z)]_{R/R_0 \text{ from } (2)} \, dz + \frac{1}{R_0^{3n+1}} \int_{d+L_d}^{d+L_d} [\phi(z)]_{R/R_0 = 1} \, dz, \quad (12)
\]

with

\[
\phi(z) = \frac{1}{(R/R_0)^{3n+1}}.
\]

The analytical evaluation of the first, second and the fourth integrals are straightforward where as the computation of third integral in the closed form in the expression for \(\psi\) equation (12) is almost a formidable task and thus will be evaluated numerically. Using the definitions from the published literature (Srivastava, 1996), the expressions for
the impedance (flow resistance), \( \lambda \), the wall shear stress in the stenotic region, \( \tau_w \) and the shear stress at stenosis throat, \( \tau_s \) are given as

\[
\lambda = \frac{\Delta p}{Q},
\]
\[
\tau_w = \frac{(-R/2)dp}{dz},
\]
\[
\tau_s = \frac{(-R/2)dp}{dz},
\]

Following now the reports of Young(1968) and Srivastava et.al.(2010a), the expression for impedance, \( \lambda \), the wall shear stress, \( \tau_w \) and shear stress at stenosis throat, \( \tau_s \), in their non-dimensional form are derived as

\[
\lambda = (1 - L_o/L) + \frac{R_0 L_o}{6n\delta L} \left[ 1 - \frac{1}{(1 - \delta/R_o)^{3n}} \right] + \frac{L_o}{2\pi L} \int_0^z \frac{d\beta}{(a + b\cos\beta)^{3n+1}},
\]
\[
\tau_w = \frac{1}{(R/R_o)^{3n}},
\]
\[
\tau_s = \frac{1}{(1 - \delta/R_o)^{3n}},
\]

where \( a = 1 - \delta/R_o \) and \( b = \delta/2R_o \), \( \beta = \pi - (2\pi/L_o)(z - d - L_o/2) \), \( \lambda = \lambda_0/\lambda_s \) and \( \tau = \tau/\tau_0 \), \( \lambda_0 \) and \( \tau_0 \) are the impedance and the shear stress for a power-law fluid in the normal artery (i.e., stenosis) and are given by

\[
\lambda_0 = \frac{2mL}{QR_0^{3n+1}} \left( \frac{3n+1}{n\pi} Q \right)^n \quad \text{and} \quad \tau_0 = \frac{m}{R_0^{3n+1}} \left( \frac{3n+1}{n\pi} Q \right)^n.
\]

It is worth to mention here that results obtained above reduce to the corresponding results for a Newtonian fluid (Young, 1968) for the artery with a composite stenosis.

**RESULTS AND DISCUSSION**

To discuss the results obtained above quantitatively, computer codes are developed to evaluate the analytical results obtained in equations (16)-(18). The various parameter values are selected (Young, 1968; Srivastava, 1996; Hershey et.al., 1964; Charm and Kurland, 1974) as

\[
R_0 \text{(cm)} = 0.01, \quad L_o/L = 1.0,5,0.1: n = 1,2/3,1/3; \quad \delta/R_0 = 0,0.05,0.10,0.15,0.20.
\]

It is to note here that the present study corresponds to a Newtonian fluid case and to the flow in normal artery for parameter value \( n = 1 \) and \( \delta/R_0 = 0 \), respectively.
The impedance (resistance to flow), $\lambda$, with the stenosis height, $\delta/R_o$, for any given value of the flow behavior index, $n$ (Fig.2). The impedance, $\lambda$, decreases with the stenosis length, $L$. 

Fig.2 Impedance, $\lambda$ versus stenosis height, $\delta/R_o$ for different, $n$.

Fig.3 Impedance, $\lambda$ versus stenosis height, $\delta/R_o$ for different, $L$. 

Numbers $n=\frac{1}{3}$, $\frac{2}{3}$, $1$, $\frac{1}{3}$

$SOLID n=1$, $DASH n=\frac{2}{3}$, $DOT n=\frac{1}{3}$
for any given value of the flow behavior index, $n$ and the stenosis height, $\delta/R_o$ (Fig.3). We observe that for any given stenosis height, $\delta/R_o$, the flow resistance, $\lambda$ assumes the maximum magnitude in Newtonian fluid ($n=1$) analysis (Fig.4).

The wall shear stress in the stenotic region, $\tau_w$ increases with the stenosis height, $\delta/R_o$ at any axial distance, $z/L_0$ (Fig.5). At any axial distance, $z/L_0$ in the stenotic region, the flow characteristic, $\tau_w$ decreases with the flow behavior index, $n$ for a given height, $\delta/R_o$ (Fig.6). The wall shear stress in the stenotic region, $\tau_w$ increases steeply in the upstream of the stenosis throat located at $z=d+L_0/2$ assumes its peak magnitude at the throat and then decreases.
rapidly in the downstream of the throat and achieves the same value at the end of the constriction profile at \( z = d + L_0 \) as its approached value at \( z = d \). One notices that there exists almost a linear relationship between the blood flow characteristic, \( \tau_w \) and the axial distance, \( z/L_0 \) in the upstream of the stenosis throat, however, the relationship between \( \tau_w \) and \( z/L_0 \) is observed to be clearly a nonlinear for any given set of parameters (Fig.5 and 6). The shear stress at the stenosis throat, \( \tau_s \) decreases with the flow behavior index \( n \) for any given stenosis height, \( \delta/R_o \) (Fig.7). Numerical results reveal that the blood flow characteristic, \( \tau_s \) increases with the stenosis size (height and length). The nature of variations of the shear stress at stenosis throat, \( \tau_s \) are similar to that of the impedance, \( \lambda \) with respect to any parameter (Fig.2 and 7).
CONCLUSIONS

A non-Newtonian (power-law) fluid has been used to discuss the response of a composite stenosis on the flow characteristics of blood. The flow resistance decreases with the non-Newtonian behavior of blood but increases with the stenosis size (height and length both). The wall shear stress at any axial distance in the stenotic region including at the stenosis throat possesses characteristics similar to that of the flow resistance with respect to any parameter.

REFERENCES